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Mathematics	Limits and Continuity	M-04

Class work Problem Solutions

1. $\lim_{x\to 0}\frac{x}{\sqrt{4-x}-\sqrt{4+x}}\times\frac{\sqrt{4-x}+\sqrt{4+x}}{\sqrt{4-x}+\sqrt{4+x}}=\lim_{x\to 0}\frac{x\left(\sqrt{4-x}+\sqrt{4+x}\right)}{4-x-4-x}$

$=\lim_{x\to 0}\frac{-x\left(\sqrt{4-x}+\sqrt{4+x}\right)}{2x}=-\frac{1}{2}\left[\sqrt{4-0}+\sqrt{4+0}\right]=\frac{-1}{2}\times 4=-2$

2. $\lim_{x\to -1}\frac{x^9-(-1)^9}{x^3-(-1)^3}=\frac{9\cdot(-1)^8}{3\cdot(-1)^2}=\frac{9}{3}=3$

3. $\lim_{x\to 3}\left[\frac{1}{x-2}+\frac{1}{x+3}\right]=\lim_{x\to 3}\left[\frac{x+3+x-2}{(x-2)(x+3)}\right]=\lim_{x\to 3}\left[\frac{2x+1}{(x-2)(x+3)}\right]=\frac{6+1}{(3-2)(3+3)}=\frac{7}{1\cdot 6}=\frac{7}{6}$

4. $\lim_{x\to -1}\frac{x^3+6x^2+12x+7}{x+1}=\lim_{x\to -1}\frac{(x+1)\left(x^2+5x+7\right)}{(x+1)}=1-5+7=3$

5. $\lim_{x\to 3}\left[\frac{1}{x-3}+\frac{9x}{27-x^3}\right]=\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{9x}{(x-3)\left(x^2+3x+9\right)}\right]=\lim_{x\to 3}\frac{x^2+3x+9-9x}{(x-3)\left(x^2+3x+9\right)}$

$=\lim_{x\to 3}\frac{x^2-6x+9}{(x-3)\left(x^2+3x+9\right)}=\lim_{x\to 3}\frac{(x-3)^2}{(x-3)\left(x^2+3x+9\right)}=\lim_{x\to 3}\frac{x-3}{x^2+3x+9}=0$

6. $\lim_{x\to 2}\frac{2x^2-5x+2}{x^2-3x+2}=\lim_{x\to 2}\frac{(x-2)(2x-1)}{(x-2)(x-1)}=\lim_{x\to 2}\frac{2x-1}{x-1}=\frac{4-1}{2-1}=3$

7. $\lim_{x\to 2}\frac{x^5\sqrt{x}-32\sqrt{2}}{x^3\sqrt{x}-8\sqrt{2}}=\lim_{x\to 2}\frac{x^{11/2}-2^{11/2}}{x^{7/2}-2^{7/2}}=\frac{\frac{11}{2}\cdot 2^{9/2}}{\frac{7}{2}\cdot 2^{5/2}}=\frac{11}{7}\cdot(2)^{4/2}=\frac{11}{7}\times 4=\frac{44}{7}$

8. $\lim_{x\to 0}\left[\frac{\sqrt{a+x}-\sqrt{a-x}}{x}\right]=\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}\cdot\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\lim_{x\to 0}\frac{2}{\sqrt{a+x}+\sqrt{a-x}}=\frac{2}{2\sqrt{a}}=\frac{1}{\sqrt{a}}$

9. $\lim_{x\to 1}\left[\frac{x+2}{x^2-5x+4}+\frac{x-4}{3\left(x^2-3x+2\right)}\right]=\lim_{x\to 1}\left[\frac{x+2}{(x-1)(x-4)}+\frac{x-4}{3(x-1)(x-2)}\right]=\lim_{x\to 1}\left[\frac{3\left(x^2-4\right)+(x-4)^2}{3(x-1)(x-2)(x-4)}\right]$

$=\frac{4}{3}\lim_{x\to 1}\frac{(x-1)^2}{(x-1)(x-2)(x-4)}=\frac{4}{3}\lim_{x\to 1}\frac{x-1}{(x-2)(x-4)}=0$

10. $\lim_{x\to a}\frac{a^x-x^a}{x^x-a^a}=\lim_{x\to a}\frac{a^x\log a-ax^{a-1}}{x^x(1+\log x)}$ (L′ Hospital Rule) $=\frac{a^a(\log a-1)}{a^a(\log a+1)}=\frac{\log a-1}{\log a+1}=-1$ (given)

$\Rightarrow \log a-1=-\log a-1 \Rightarrow 2\log a=0 \Rightarrow \log a=0 \Rightarrow a=1$

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11. $\lim\limits_{\theta\to 0}\dfrac{\left(\dfrac{\sin 2\theta}{2\theta}\right)2\theta\cdot\left(\dfrac{\sin 3\theta}{3\theta}\right)3\theta}{3\theta\cdot\left(\dfrac{\tan 4\theta}{4\theta}\right)4\theta}=\dfrac{1\cdot 2\cdot 3}{3\cdot 1\cdot 4}=\dfrac{1}{2}$

12. $\lim\limits_{\theta\to 0}\dfrac{\cos 5\theta-\cos 7\theta}{\theta^2}=\lim\limits_{\theta\to 0}\dfrac{-2\sin 6\theta\cdot\sin(-\theta)}{\theta^2}=\lim\limits_{\theta\to 0}\dfrac{2\sin 6\theta\cdot\sin\theta}{\theta^2}=\lim\limits_{\theta\to 0}\dfrac{2\cdot\sin 6\theta}{6\theta}\times 6\cdot\dfrac{\sin\theta}{\theta}=2\cdot 1\cdot 6\cdot 1=12$

13. $\lim\limits_{x\to 2}\dfrac{\sin\dfrac{\pi x}{2}}{2x-1}=\dfrac{\sin\pi\cdot\dfrac{2}{2}}{2\cdot 2-1}=\dfrac{0}{4-1}=0$

14. $\lim\limits_{x\to 0}\dfrac{3\tan x-8x}{3x+\sin 2x}=\lim\limits_{x\to 0}\dfrac{x\left[\dfrac{3\tan x}{x}-8\right]}{x\left[3+\dfrac{\sin 2x}{2x}\cdot 2\right]}=\dfrac{3-8}{3+2}=\dfrac{-5}{5}=-1$

15. $\lim\limits_{x\to a}\dfrac{\sin^2 x-\sin^2 a}{x^2-a^2}=\lim\limits_{x\to a}\dfrac{\sin(x+a)\cdot\sin(x-a)}{x^2-a^2}$

$=\lim\limits_{x\to a}\dfrac{\sin(x+a)\cdot\sin(x-a)}{(x+a)(x-a)}=\dfrac{\sin 2a}{2a}\cdot 1=\dfrac{2\sin a\cdot\cos a}{2a}=\dfrac{\sin a\cdot\cos a}{a}$

16. $\lim\limits_{\theta\to\frac{\pi}{4}}\dfrac{\sqrt{2}-\sqrt{2}\left\{\dfrac{1}{\sqrt{2}}\cos\theta+\dfrac{1}{\sqrt{2}}\sin\theta\right\}}{\left[4\left(\theta-\dfrac{\pi}{4}\right)\right]^2}=\lim\limits_{\theta\to\frac{\pi}{4}}\dfrac{\sqrt{2}-\sqrt{2}\left\{\cos\left(\theta-\dfrac{\pi}{4}\right)\right\}}{16\left(\theta-\dfrac{\pi}{4}\right)^2}=\lim\limits_{\theta\to\frac{\pi}{4}}\dfrac{\sqrt{2}\cdot 2\sin^2\left(\dfrac{\theta-\frac{\pi}{4}}{2}\right)}{16\cdot\dfrac{\left(\theta-\frac{\pi}{4}\right)^2}{4}\cdot 4}=\dfrac{1\cdot\sqrt{2}}{2\cdot 16}=\dfrac{1}{16\sqrt{2}}$

17. $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{-\cos 8x}=\lim\limits_{x\to 0}\dfrac{5\sin 5x}{8\sin 8x}$ (by L′ Hospitals rule) $=\lim\limits_{x\to 0}\dfrac{5}{8}\cdot\dfrac{\dfrac{\sin 5x}{5x}\cdot 5x}{\dfrac{\sin 8x}{5x}\cdot 8x}=\dfrac{25}{64}$

18. $\lim\limits_{x\to\infty}\left[\dfrac{x+7}{x+1}\right]^x\left[\dfrac{x+7}{x+1}\right]^4=\lim\limits_{x\to\infty}\dfrac{x^x\left(1+\dfrac{7}{x}\right)^x}{x^x\left(1+\dfrac{1}{x}\right)^x}.\left[\dfrac{x\left(1+\dfrac{7}{x}\right)}{x\left(1+\dfrac{1}{x}\right)}\right]^4=\lim\limits_{x\to\infty}\dfrac{\left[(1+7/x)^{x/7}\right]^7}{\left(1+\dfrac{1}{x}\right)^x}.\left(\dfrac{1+7/x}{1+1/x}\right)^4=\dfrac{e^7}{e}.\left(\dfrac{1}{1}\right)^4=e^6$

19. $\lim\limits_{x\to 0}\left(\dfrac{1+7x^2}{1+2x^2}\right)^{1/x^2}=\lim\limits_{x\to 0}\dfrac{\left[\left(1+7x^2\right)^{1/7x^2}\right]^7}{\left[\left(1+2x^2\right)^{1/2x^2}\right]^2}=\dfrac{e^7}{e^2}=e^5\qquad\left[\because\lim\limits_{x\to 0}(1+x)^{1/x}=e\right]$

20. limit $=\lim\limits_{x\to\frac{\pi}{4}}(2-\tan x)^{\log\tan x}=(2-1)^{\log 1}=1^{\circ}=1$

21. $\lim\limits_{x\to 0}\dfrac{10^x-5^x+2^x-1}{x^2}=\lim\limits_{x\to 0}\dfrac{2^x.5^x-5^x-2^x+1}{x^2}=\lim\limits_{x\to 0}\dfrac{\left(2^x-1\right)}{x}\dfrac{\left(5^x-1\right)}{x}=\log 2.\log 5$

22. $\lim\limits_{n\to\infty}\dfrac{1^3+2^3+3^3+........+n^3}{n^2(n+1)(2n+3)}=\lim\limits_{n\to\infty}\dfrac{n^2(n+1)^2/4}{n^2(n+1)(2n+3)}=\lim\limits_{n\to\infty}\dfrac{(n+1)}{4(2n+3)}=\lim\limits_{n\to\infty}\dfrac{n(1+1/n)}{4n(2+3/n)}=\dfrac{1}{4.2}=\dfrac{1}{8}$

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23. limit $= \lim_{n\to\infty} \dfrac{\sin^2(n!)}{n^{1-p}\left(1+\frac{1}{n}\right)} = \dfrac{\text{some number between } 0 \,\&\, 1}{\infty} = 0$

24. Limit $= \lim_{x\to\infty} \dfrac{x+\sqrt{x}-x}{\sqrt{x+\sqrt{x}}+\sqrt{x}} = \lim_{x\to\infty} \dfrac{\sqrt{x}}{\sqrt{x}\left\{\sqrt{1+\frac{1}{\sqrt{x}}}+1\right\}} = \dfrac{1}{2}$

25. $\lim_{x\to\infty}\left(1-\dfrac{4}{x-1}\right)^{3x-1} = \lim_{x\to\infty}\left[\left(1-\dfrac{4}{x-1}\right)^{\frac{x-1}{4}}\right]^{-\frac{4(3x-1)}{(x-1)}} \quad \left[\because \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e\right] = \lim_{n\to\infty}(e)^{-4\cdot\left(\frac{3-1/n}{1-1/n}\right)} = e^{-4\left(\frac{3-0}{1-0}\right)} = e^{-12}$

26. $\lim_{x\to\infty} \dfrac{\left(\sqrt{a^2x^2+bx+c}-ax\right)\left(\sqrt{a^2x^2+bx+c}+ax\right)}{\sqrt{a^2x^2+bx+c}+ax}$

$= \lim_{x\to\infty} \dfrac{a^2x^2+bx+c-a^2x^2}{\sqrt{a^2x^2+bx+c}+ax} = \lim_{x\to\infty}\left[\dfrac{x\left(b+\frac{c}{x}\right)}{x\left[\sqrt{a^2+\frac{b}{x}+\frac{c}{x^2}}+a\right]}\right] = \dfrac{b}{2a}$

27. $\lim_{x\to\infty} \sqrt{\dfrac{x-\sin x}{x+\cos^2 x}} = \lim_{x\to\infty} \sqrt{\dfrac{1-\sin x/x}{1+\left(\cos^2 x\right)/x}} = \sqrt{\dfrac{1-0}{1+0}} = 1$

28. $\lim_{n\to\infty}(0.2)^{\log_{\sqrt{5}}\left[\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots..n\ terms\right]} = (0.2)^{\log_{\sqrt{5}}\left[\frac{1/4}{1-\frac{1}{2}}\right]} = (0.2)^{\log_{\sqrt{5}}\left(\frac{1}{2}\right)} = (0.2)^{2\log_5\frac{1}{2}} = (0.2)^{\log_5\frac{1}{4}} = (0.2)^{-\log_5 4}$

$= \left(\dfrac{2}{10}\right)^{-\log_5 4} = \left(5^{-1}\right)^{-\log_5 4} = 5^{\log_5 4} = 4$

29. Using $\lim_{x\to\infty}\left(\dfrac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right)^x = \sqrt[3]{abc}$, we get $\lim_{x\to\infty}\left(\dfrac{2^{1/x}+27^{1/x}+8^{1/x}}{3}\right) = (2\cdot 27\cdot 8)^{1/3} = 6\cdot(2)^{1/3}$

30. $\lim_{x\to 0}\left[\dfrac{1+\tan x}{1+\sin x}\right]^{\operatorname{cosec} x} = e^{\lim_{x\to 0}\left[\frac{1+\tan x}{1+\sin x}-1\right]\operatorname{cosec} x} = e^{\lim_{x\to 0}\left[\frac{1+\tan x-1-\sin x}{1+\sin x}\right]\frac{1}{\sin x}} = e^{\lim_{x\to 0}\left[\frac{\frac{\sin x}{\cos x}-\sin x}{1+\sin x}\right]\cdot\frac{1}{\sin x}}$

$= e^{\lim_{x\to 0}\left[\frac{\sin x\left(\frac{1-\cos x}{\cos x}\right)}{1+\sin x}\right]\cdot\frac{1}{\sin x}} = e^{\left[\frac{1-1}{1(1+0)}\right]} = e^0 = 1$

31. $\lim_{x\to 0} f(x) = e$, $f(0) = e$ $\therefore$ $f(x)$ is continuous at $x=0$

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32. $\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot 2=2$, $f(0)=k=2$ ($\because$ $f(x)$ is continuous at $x=0$)

33. $f(x)=|x|+|x-1|$ For $x<0$, $0\le x<1$ and $x\ge 1$, $f(x)=\begin{cases}-2x+1, & x<0\\ 1, & 0\le x<1\\ 2x-1, & x\ge 1\end{cases}$

It is continuous at $x=0$ and $x=1$

34. $\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{1-\cos 4x}{x^2}=\lim_{x\to 0}\frac{4\sin 4x}{2x}=\lim_{x\to 0}\frac{8.\sin 4x}{4x}=8$

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}\frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}=\lim_{x\to 0}\frac{\sqrt{x}\left(\sqrt{16+\sqrt{x}}+4\right)}{16+\sqrt{x}-16}=\lim_{x\to 0}\left(\sqrt{16+\sqrt{x}}+4\right)=4+4=8$

$\therefore \lim_{x\to 0} f(x)=8$ $\therefore$ $a=8$. $\left[\because \lim_{x\to 0} f(x)=f(0)=a\right]$

35. $\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\sin 5x}{x^2+2x}=\lim_{x\to 0}\frac{5\cos 5x}{2x+2}$ $\left[\because\ f(0)=k+\frac{1}{2}\right]=\frac{5}{2}=k+\frac{1}{2}$ $\therefore$ $k=\frac{5}{2}-\frac{1}{2}=2$

36. Given limit $=\lim_{n\to\infty}\frac{(n+2)(n+1)!+(n+1)!}{(n+2)(n+1)!-(n+1)!}$

$=\lim_{n\to\infty}\frac{(n+2+1)}{(n+2-1)}=\lim_{n\to\infty}\frac{n+3}{n+1}=\lim_{n\to\infty}\frac{1+\frac{3}{n}}{1-\frac{2}{n}}=\frac{1+0}{1-0}=1$

37. $\lim_{x\to 0}\frac{2f(x)-3f(2x)+f(4x)}{x^2}$ (By L-H rule)

$=\lim_{x\to 0}\frac{2f'(x)-6f'(2x)+4f'(4x)}{2x}=\lim_{x\to 0}\frac{2f''(x)-12f''(2x)+16f''(4x)}{2}$

$=\frac{2f''(0)-12f''(0)+16f''(0)}{2}=\frac{6f''(0)}{2}=3f''(0)=3(4)=12$

Homework Problem Solutions

1. $\lim_{x\to 3}\frac{x^2+2x-15}{x^2-5x+6}=\lim_{x\to 3}\frac{(x+5)(x-3)}{(x-2)(x-3)}=\lim_{x\to 3}\frac{x+5}{x-2}=\frac{3+5}{3-2}=\frac{8}{1}=8$

2. $\lim_{x\to a}\frac{x^7-a^7}{x^5-a^5}=\frac{7}{5}a^{7-5}=\frac{7}{5}a^2$

3. $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x}=\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x}\cdot\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$

$$= \lim_{x\to 0}\frac{\left(1+x^2\right)-\left(1-x^2\right)}{\left[\sqrt{1+x^2}+\sqrt{1-x^2}\right]x} = \lim_{x\to 0}\frac{2x}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \frac{0}{1+1} = 0$$

4. $$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3-x}}{x} = \lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3-x}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3-x}}{\sqrt{3+x}+\sqrt{3-x}}$$

$$= \lim_{x\to 0}\frac{(3+x)-(3-x)}{\left(\sqrt{3+x}+\sqrt{3-x}\right)} = \lim_{x\to 0}\frac{2x}{x\left(\sqrt{3+x}+\sqrt{3-x}\right)} = \lim_{x\to 0}\frac{2}{\sqrt{3+x}+\sqrt{3-x}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

5. $$\lim_{x\to 3}\frac{(x-3)\left(x^2-2\right)}{x^2-2x-3} = \lim_{x\to 3}\frac{(x-3)\left(x^2-2\right)}{(x-3)(x+1)} = \lim_{x\to 3}\frac{x^2-2}{x+1} = \frac{9-2}{3+1} = \frac{7}{4}$$

6. $$\lim_{x\to 0}\frac{\sqrt{1+x+x^3}-1}{x} = \lim_{x\to 0}\frac{\left(\sqrt{1+x+x^3}-1\right)\left(\sqrt{1+x+x^3}+1\right)}{x\left(\sqrt{1+x+x^3}+1\right)}$$

$$= \lim_{x\to 0}\frac{1+x+x^3-1}{x\left(\sqrt{1+x+x^3}+1\right)} = \lim_{x\to 0}\frac{x\left(1+x^2\right)}{x\left(\sqrt{1+x+x^3}+1\right)} = \lim_{x\to 0}\frac{1+x^2}{\sqrt{1+x+x^3}+1} = \frac{1+0}{1+1} = \frac{1}{2}$$

7. $$\lim_{x\to 2}\frac{\left(x^2+5\right)^{1/2}-3}{x-2} = \lim_{x\to 2}\frac{x^2+5-9}{(x-2)\left[\sqrt{x^2+5}+3\right]} = \lim_{x\to 2}\frac{x^2-4}{(x-2)\left[\sqrt{x^2+5}+3\right]}$$

$$= \lim_{x\to 2}\frac{x+2}{\sqrt{x^2+5}+3} = \frac{2+2}{\sqrt{4+5}+3} = \frac{4}{3+3} = \frac{4}{6} = \frac{2}{3}$$

8. $$\lim_{x\to 0}\frac{\sin x^\circ}{x} = \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\times\frac{\pi}{180} = 1\cdot\frac{\pi}{180} = \frac{\pi}{180}$$

9. $$\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{4}\right)}{\frac{x^2}{16}\times 16} = 1\cdot\frac{1}{16} = \frac{1}{16}$$

10. $$\lim_{x\to 0}\frac{\sin^2\frac{x}{3}}{x^2} = \frac{\lim}{x\to 0}\frac{\sin^2\frac{x}{3}}{\frac{x^2}{9}\times 9} = 1\cdot\frac{1}{9} = \frac{1}{9}$$

11. Put $x-1=t \Rightarrow x=1+t$ as $x\to 1,\ t\to 0$

$$\lim_{t\to 0}\frac{\sin(\pi+\pi t)}{t} = \lim_{t\to 0}\frac{-\sin\pi t}{\pi t}\cdot\pi = -1\cdot\pi = -\pi$$

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12. $\lim\limits_{x\to 0}\dfrac{\tan\left[\sin^{-1}3x\right]}{\sin^{-1}(2\tan 2x)}=\lim\limits_{x\to 0}\left[\dfrac{\dfrac{\tan\left[\sin^{-1}3x\right]}{\sin^{-1}3x}\cdot\dfrac{\sin^{-1}3x}{3x}\cdot 3x}{\dfrac{\sin^{-1}(2\tan 2x)}{2\tan 2x}\cdot\dfrac{2\tan 2x}{2x}\cdot 2x}\right]=\dfrac{1\cdot 1\cdot 3}{1\cdot 1\cdot 2}=\dfrac{3}{2}$

13. $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta\cos\theta}=\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta}\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta}=\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta}\left[\dfrac{0}{0}\ form\right]=\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{1}=0$

14. $\lim\limits_{x\to\frac{\pi}{2}}(\sec x-\tan x)=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{\cos x}=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{-\cos x}{-\sin x}=0$

15. $\lim\limits_{x\to\pi}\dfrac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$ Put $x=\pi+h$, where $x\to\pi, h\to 0$

$=\lim\limits_{h\to 0}\dfrac{\sqrt{2-\cos h}-1}{h^2}=\lim\limits_{h\to 0}\dfrac{[(2-\cos h)-1]}{h^2\left[\sqrt{2-\cos h}+1\right]}=\lim\limits_{h\to 0}\dfrac{1-\cos h}{h^2\left[\sqrt{2-\cos h}+1\right]}=\lim\limits_{h\to 0}\dfrac{2\sin^2 h/2}{h^2\left[\sqrt{2-\cos h}+1\right]}$

$=\lim\limits_{h\to 0}\left(\dfrac{\sin h/2}{h/2}\right)^2\cdot\dfrac{1}{2}\dfrac{1}{\sqrt{2-\cos h}+1}=\dfrac{1}{2}\cdot\dfrac{1}{(1+1)}=\dfrac{1}{4}$

16. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{\sec^2 x-\cos x}{3x^2}$ (by L′ Hospitals rule)

$=\lim\limits_{x\to 0}\dfrac{1+\tan^2 x-\cos x}{3x^2}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}+\lim\limits_{x\to 0}\dfrac{\tan^2 x}{3x^2}$

$=\lim\limits_{x\to 0}\dfrac{\sin x}{6x}+\dfrac{1}{3}(1)$ (by L′ Hospitals rule) $=\dfrac{1}{6}+\dfrac{1}{3}=\dfrac{1+2}{6}=\dfrac{3}{6}=\dfrac{1}{2}$.

17. $\lim\limits_{x\to 0}\dfrac{\log\cos x}{x}=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{\cos x}\cdot\sin x}{1}=0$ (By L′ Hospital's Rule)

18. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\times\dfrac{1-\sin x}{(\pi-2x)^3}=\lim\limits_{x\to\frac{\pi}{2}}\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)\cdot\dfrac{1-\sin x}{8\left(\dfrac{\pi}{2}-x\right)^3}$

Put $\dfrac{\pi}{2}-x=h$ where $x\to\dfrac{\pi}{2}, h\to 0$ $\quad\lim\limits_{h\to 0}\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\left(\dfrac{\pi}{2}-h\right)\right)\dfrac{1-\sin\left(\dfrac{\pi}{2}-h\right)}{8\left(h^3\right)}$

$=\lim\limits_{h\to 0}\tan\left(\dfrac{h}{2}\right)\dfrac{1-\cos h}{8h^3}=\lim\limits_{h\to 0}\dfrac{\tan\left(\dfrac{h}{2}\right)}{\dfrac{h}{2}\cdot 2}\cdot\dfrac{2\sin^2 h/2}{4\cdot\left(\dfrac{h}{2}\right)^2\cdot 8}=1\cdot 1\cdot\dfrac{1}{4\cdot 8}=\dfrac{1}{32}$

19. Using L' Hospital's rule, we get $\lim_{x\to 0}\frac{3^x\log 3-2^x\log 2}{\cos x}=\frac{\log 3-\log 2}{1}=\log\frac{3}{2}$

20. limit $=\lim_{x\to 0}\left(1+2.\frac{x^2}{1+3x^2}\right)^{1/x^2}=\lim_{x\to 0}\left[1+2.\frac{1}{3+\frac{1}{x^2}}\right]^{\left(3+\frac{1}{x^2}\right).\frac{\frac{1}{x^2}}{3+\frac{1}{x^2}}}$

Put $\frac{1}{3+\frac{1}{x^2}}=y.$ as $x\to 0,\ \ y\to 0\ \Rightarrow$ limit $=\left\{\lim_{y\to 0}\left(1+\frac{2}{y}\right)^y\right\}^{\lim_{x\to 0}\frac{\frac{1}{x^2}}{3+\frac{1}{x^2}}}=e^{2\lim_{x\to 0}\frac{1}{3x^2+1}}=e^2$

21. $\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{3n^2+5}=\lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)}{2.n^2\left(3+\frac{5}{n^2}\right)}=\frac{1}{2.3}=\frac{1}{6}$

22. $\lim_{x\to 0}(1+ax)^{b/x}=\lim_{x\to 0}\left[(1+ax)^{\frac{1}{ax}}\right]^{\frac{ax\cdot b}{x}}=\lim_{x\to 0}\left[(1+ax)^{\frac{1}{ax}}\right]^{ab}=e^{ab}$

23. $\lim_{n\to\infty}\frac{1^2+2^2+3^3+\ldots+n^2}{n^3}=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}=\frac{1}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\frac{2}{6}=\frac{1}{3}$

24. $\lim_{n\to\infty}\frac{1+2+\ldots+n}{n^2+100}=\lim_{n\to\infty}\frac{n(n+1)}{2\left(n^2+100\right)}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2\left(1+\frac{100}{n^2}\right)}=\frac{1}{2}$

25. $\lim_{n\to\infty}\left(3^n+4^n\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left(4^n\right)^{\frac{1}{n}}\left[1+\left(\frac{3}{4}\right)^n\right]^{\frac{1}{n}}=4\lim_{n\to\infty}\left[1+\left(\frac{3}{4}\right)^n\right]^{\frac{1}{n}}=4(1+0)=4$

26. $\lim_{x\to 1}(\log_2 2x)^{\frac{1}{\log_2 x}}=\lim_{x\to 1}(\log_2 2+\log_2 x)^{\frac{1}{\log_2 x}}=\lim_{x\to 1}(1+\log_2 x)^{\frac{1}{\log_2 x}}=\lim_{y\to 0}(1+y)^{1/y}=e$

$\left[\text{ If } y=\log_2 x \text{ as } x\to 1, y\to 0\right]$

27. Since $|x|=\begin{cases}-x & x<0\\ x & x\geq 0\end{cases}$ we have

$LHL=RHL$ only at $x=0$ and at other points $LHL\neq RHL$

28. $\lim_{x\to -2}\left(2x^2+5x+2\right)=2\cdot 4+5(-2)+2=8-10+2=0$

For $f(x)$ to be continuous $LHL=RHL=f(a)$

$\therefore\ \ f(-2)=k\Rightarrow k=0$

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29. $f(x)=\dfrac{x-4}{x-1}$ is continuous at $x=0,2,4$

30. $f(x)=\sin x+\log_e x$, x is continuous $\forall x>0$

($\because$ $\log$ of –ve value is not defined)

31. If $\lim\limits_{x\to a} f(x)$ exists but $\neq f(a)$,then $f(x)$ is not continuous at $x=a$

32. $\lim\limits_{h\to 0} f(0+h)=-\dfrac{1}{2}$ and $f(0)=-\dfrac{1}{2}$

$\lim\limits_{h\to 0} f(0-h)=p$ since $f(x)$ is continuous in $[-1,1]$ then $p=-\dfrac{1}{2}$

33. $f(x)$ is continuous at none of $x=1$ and $x=-1$

34. $\lim\limits_{x\to 2}\dfrac{x^5-32}{x-2}=5(2)^4=80$ $\left[\because \lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}\right]$

Since $f(x)$ is continuous at $x=2$. $\therefore \lim\limits_{x\to 2} f(x)=f(2)\Rightarrow 80=k$

35. $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=\lim\limits_{x\to 0}\dfrac{\sin x}{1}=0$

Since $f(x)$ is continuous at $x=0$. $\therefore$ $\lim\limits_{x\to 0} f(x)=f(0)\Rightarrow 0=k$ $\therefore k=0$

36. For continuity at $x=1$, $R=L=V\Rightarrow \dfrac{1}{a}=a$

$\therefore$ $a=1,-1$. For continuity at $x=\sqrt{2}$

$R=L=V\Rightarrow a=b^2-2b$ when $a=+1, b^2-2b-1=0$

$\therefore$ $b=1\pm\sqrt{2}$. These values are not given. When $a=-1, b^2-2b+1=0$. $\therefore$ $b=1$

37. $\lim\limits_{x\to 0}\log_{\cos x/2}\cos x=\lim\limits_{x\to 0}\dfrac{\log\cos x}{\log\cos x/2}=\lim\limits_{x\to 0}\dfrac{\frac{1}{\cos x}.-\sin x}{\frac{1}{\cos\frac{x}{2}}.-\sin\frac{x}{2}.\frac{1}{2}}=\lim\limits_{x\to 0}\dfrac{2\sin x}{\sin\frac{x}{2}}$ (by L′ Hospitals rule)

$=\lim\limits_{x\to 0}\dfrac{2.2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}=4$ $\lim\limits_{x\to 0}4\cos\frac{x}{2}=4\cdot 1=4$