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Mathematics	Limits and Continuity	M-04

Limits and Continuity

A function $f(x)$ is said to have a limit l as x tends to a, if the absolute difference $|f(x)-l|$ can be made as small as possible whenever the absolute difference $|x-a|$ is sufficiently small.

Symbolically denoted as $\lim_{x\to a} f(x) = l$

Important Limits

1. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
2. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{x\to 0} \frac{\sin^{-1}(x)}{x} = 1$
3. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$ and $\lim_{x\to 0} \frac{\tan^{-1}(x)}{x} = 1$
4. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$
5. $\lim_{n\to 0} (1+n)^{1/n} = e$
6. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$, where $a > 0$
7. $\lim_{x\to 0} \frac{e^x - 1}{x} = \log_e e = 1$
8. $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

Left hand Limit (LHL) & Right hand limit (RHL)

Let $f(x)$ be a real value of function defined in a neighborhood of a except possibly at a then $f(x)$ is said to have a left hand limit l as x tends to a from the left for $\in > 0$ however small there exists a positive number δ such that $|f(x)-l| < \in$ wherever $a-\delta < x < a$ and is denoted by $\lim_{x\to a^-} f(x) = l$ - right hand limit (RHL).

Similarly, $|f(x)-l| < \in$ wherever $a < x < a+\delta$ and is denoted by $\lim_{x\to a^+} f(x) = l$ - left hand limit (LHL)

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Limit of a function exists if LHL = RHL

Continuity

A function $f(x)$ is said to be continuous at $x=a$, if $\lim_{x\to a} f(x)=f(a)$

i.e. $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=f(a)$

Continuity in a interval

A function $f(x)$ is said to be continuous in a interval I (open or closed) if it is continuous at every point of the interval.

Time saving results (Short cut methods)

- $\lim_{n\to\infty} n\left(\sqrt{n^2-a}-n\right)=\frac{a}{2}$
- $\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}=\frac{1}{\sqrt{a}}$
- $\lim_{x\to 0}\frac{\sqrt{1+x^n}-\sqrt{1-x^n}}{x^n}=1$
- $\lim_{x\to\infty}\frac{a^{x+\alpha}+b}{a^{x+\beta}+c}=a^{\alpha-\beta}$
- $\lim_{x\to 0}\frac{\sqrt[n]{a+x^m}-\sqrt[n]{a-x^m}}{x^m}=\frac{2}{n}.a^{\left(\frac{1}{n}-1\right)}$
- $\lim_{x\to\infty}\left\{\sqrt{x^2+ax+b}-x\right\}=\frac{a}{2}$
- Let F be a set of functions defined as $F=\left\{x, \sin x, \tan x, \sinh x, \tanh x, \sin^{-1}x, \cos^{-1}x, \sinh^{-1}x, \tanh^{-1}x\right\}$

  (i) If $f(x), g(x)\in F$, then $\lim_{x\to 0}\frac{f(mx)}{g(nx)}=\frac{m}{n}$

  (ii) If $f_1(x), f_2(x), g_1(x), g_2(x)\in F$ then $\lim_{x\to 0}\frac{f_1(ax)\pm f_2(bx)}{g_1(cx)\pm g_2(dx)}=\frac{a\pm b}{c\pm d}$.

  (iii) If $f_1(x), f_2(x), g_1(x), g_2(x)\in F$ & $m+n=p+q$ then $\lim_{x\to 0}\frac{f_1^m(ax)\cdot f_2^n(bx)}{g_1^p(cx)\cdot g_2^q(dx)}=\frac{a^m\cdot b^n}{c^p\cdot d^q}$

  (iv) If $f(x)\in F$ then $\lim_{x\to 0}\frac{1-\cos ax}{f(bx)}=0$.

  (v) If $f_1(x), f_2(x)\in F$ then $\lim_{x\to 0}\frac{\cos ax-\cos bx}{f_1(cx).f_2(dx)}=\frac{b^2-a^2}{2cd}$.

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(vi) If $f_1(x), f_2(x) \in F$ then $\lim_{x\to 0} \frac{\cos^n ax - \cos^n bx}{f_1(cx).f_2(dx)} = \frac{n(b^2-a^2)}{2cd}$.

(vii) If $f(x) \in F$ then $\lim_{x\to 0} \frac{\tan^n ax - \tan^n bx}{[f(x)]^{n+2}} = \frac{n.a^{n+2}}{2}$.

- $\lim_{x\to 0} \frac{1-\cos nx}{1-\cos mx} = \frac{n^2}{m^2}$, $\lim_{x\to 0} \frac{\sin ax - \sin bx}{mx} = \frac{a-b}{m}$.
- $\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a}\{g(x)[f(x)-1]\}}$, provided $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \infty$
- If $f(x+y) = f(x).f(y)$, $\forall\, x, y$ & $f(k) = l$ & $f'(0) = m$ then $f'(k) = lm$

Class work Problems

1. $\lim_{x\to 0} \frac{x}{\sqrt{4-x} - \sqrt{4+x}}$ is

(a) 2 (b) -2 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

2. $\lim_{x\to -1} \frac{x^9+1}{x^3+1} =$

(a) $\frac{1}{3}$ (b) 3 (c) 2 (d) 1

3. $\lim_{x\to 3} \left[\frac{1}{x-2} + \frac{1}{x+3}\right] =$

(a) $\frac{7}{6}$ (b) 5 (c) $\frac{1}{4}$ (d) $\frac{3}{7}$

4. $\lim_{x\to -1} \frac{x^3+6x^2+12x+7}{x+1} =$

(a) ∞ (b) 0 (c) 3 (d) 1

5. $\lim_{x\to 3} \left[\frac{1}{x-3} + \frac{9x}{27-x^3}\right] =$

(a) 1 (b) 0 (c) 2 (d) $\frac{1}{2}$

6. $\lim_{x\to 2} \frac{2x^2-5x+2}{x^2-3x+2} =$

(a) 1 (b) 2 (c) 3 (d) 8

7. $\lim_{x\to 2} \frac{x^2\sqrt{x} - 32\sqrt{2}}{x^3\sqrt{x} - 8\sqrt{2}} =$

(a) $\frac{22}{7}$ (b) $\frac{44}{7}$ (c) $\frac{41}{7}$ (d) $\frac{48}{7}$

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8. The value of $\lim_{x\to 0}\left[\frac{\sqrt{a+x}-\sqrt{a-x}}{x}\right]$ is

(a) $\sqrt{a}$ (b) $\frac{1}{\sqrt{a}}$ (c) 0 (d) $\frac{1}{2}$

9. $\lim_{x\to 1}\left[\frac{x+2}{x^2-5x+4}+\frac{x-4}{3\left(x^2-3x+2\right)}\right]=$

(a) 0 (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) 1

10. $\lim_{x\to a}\frac{a^x-x^a}{x^x-a^a}=-1$, then

(a) $a=1$ (b) $a=0$ (c) $a=e$ (d) none of these

11. $\lim_{\theta\to 0}\frac{\sin 2\theta\cdot\sin 3\theta}{3\theta\cdot\tan 4\theta}$ is

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 2 (d) $\frac{1}{2}$

12. $\lim_{\theta\to 0}\frac{\cos 5\theta-\cos 7\theta}{\theta^2}=$

(a) $\frac{1}{6}$ (b) $\frac{1}{12}$ (c) 12 (d) 6

13. $\lim_{x\to 2}\frac{\sin \pi\frac{x}{2}}{2x-1}$ is

(a) 1 (b) 0 (c) π (d) ∞

14. $\lim_{x\to 0}\frac{3\tan x-8x}{3x+\sin 2x}=$

(a) -1 (b) 1 (c) 0 (d) none of these

15. $\lim_{x\to a}\frac{\sin^2 x-\sin^2 a}{x^2-a^2}=$

(a) 1 (b) $\cos a$ (c) $\frac{\sin a\cdot\cos a}{a}$ (d) none of these

16. $\lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}=$

(a) $\frac{1}{16}$ (b) $\frac{1}{16\sqrt{2}}$ (c) $\frac{1}{8}$ (d) $\frac{1}{8\sqrt{2}}$

17. $\lim_{x\to 0}\frac{1-\cos 5x}{1-\cos 8x}=$

(a) $\frac{25}{64}$ (b) $\frac{64}{25}$ (c) $\frac{5}{8}$ (d) $\frac{15}{23}$

18. $\lim_{x\to\infty}\left[\frac{x+7}{x+1}\right]^{x+4} =$

(a) e^{10} (b) e^{6} (c) $\sqrt[6]{e}$ (d) $e^{2/3}$

19. $\lim_{x\to 0}\left(\frac{1+7x^2}{1+2x^2}\right)^{1/x^2} =$

(a) e^2 (b) e^3 (c) e^4 (d) e^5

20. $\lim_{x\to\pi/4}(2-\tan x)^{\log\tan x}$ is equal to

(a) 0 (b) 1 (c) e (d) e^{-1}

21. $\lim_{x\to 0}\frac{10^x-5^x-2^x+1}{x^2} =$

(a) $\log 2.\log 5$ (b) $\log 10$ (c) $\log_5{}^2$ (d) none of these

22. $\lim_{n\to\infty}\frac{1^3+2^3+3^3+.........n^3}{n^2(n+1)(2n+3)} =$

(a) 1/2 (b) 1/4 (c) 1/8 (d) none of these

23. $\lim_{n\to\infty}\frac{n^p\sin^2(n!)}{n+1},\quad 0<p<1$, is equal to

(a) 0 (b) ∞ (c) 1 (d) none of these

24. $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)$ is equal to

(a) 1 (b) 0 (c) 1/2 (d) none of these

25. $\lim_{x\to\infty}\left(1-\frac{4}{x-1}\right)^{3x-1}$

(a) e^{12} (b) e^{-12} (c) e^4 (d) e^3

26. $\lim_{x\to\infty}\sqrt{a^2x^2+bx+c}-ax$ is

(a) $\frac{b}{2a}$ (b) $\frac{b}{a}$ (c) 0 (d) $\frac{2b}{a}$

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27. If $f(x)=\sqrt{\left(\frac{x-\sin x}{x+\cos^2 x}\right)}$, then $\lim_{x\to\infty} f(x)$ is

(a) 0 (b) ∞ (c) 1 (d) none of these

28. $\lim_{n\to\infty}(0.2)^{\log_{\sqrt{5}}(1/4+1/8+1/16+\ldots\text{to n terms})}$ is equal to

(a) 2 (b) 4 (c) 8 (d) 0

29. $\lim_{x\to\infty}\left(\frac{2^{1/x}+27^{1/x}+8^{1/x}}{3}\right)^x =$

(a) $3\cdot(2)^{1/3}$ (b) $2\cdot(2)^{1/3}$ (c) $4\cdot(2)^{1/3}$ (d) $6\cdot(2)^{1/3}$

30. $\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x} =$

(a) e (b) e^{-1} (c) 1 (d) none of these

31. $f(x)=\begin{cases}(1+x)^{1/x} & \forall\, x\neq 0\\ e & \forall\, x=0\end{cases}$, then $f(x)$ is

(a) continuous at $x=0$ (b) discontinuous at $x=0$

(c) continuous at $x=e$ (d) None of these

32. If $f(x)=\begin{cases}\frac{\sin 2x}{x} & for \quad x\neq 0\\ k \quad for & x=0\end{cases}$ is continuous at $x=0$, then $k=$ ____________

(a) 1 (b) 2 (c) $-\frac{1}{2}$ (d) -1

33. Let $f(x)=|x|+|x-1|$ then

(a) $f(x)$ is continuous at $x=0$ as well as at $x=1$

(b) $f(x)$ is continuous at $x=0$ but not at $x=1$

(c) $f(x)$ is continuous at $x=1$, but not at $x=0$ (d) None of these

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34. If $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & x<0\\ a\ , & x=0\\ \dfrac{x}{\sqrt{16+\sqrt{x}}-4}, & x>0\end{cases}$ then the value of a for which $f(x)$ is continuous at $x=0$ is

(a) 5 (b) 8 (c) 4 (d) 3

35. If $f(x)=\begin{cases}\dfrac{\sin 5x}{x^2+2x}, & x\neq 0\\ k+\dfrac{1}{2}, & x=0\end{cases}$ is continuous at $x=0$, then the value of k is

(a) 1 (b) -2 (c) 2 (d) $\frac{1}{2}$

36. $\lim_{n\to\infty}\dfrac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}=$

(a) 0 (b) 1 (c) -1 (d) none of these

37. Let $f'(x)$ be continuous at $x=0$ and $f''(0)=4$. The value of $\lim_{x\to 0}\dfrac{2f(x)-3f(2x)+f(4x)}{x^2}$ is

(a) 11 (b) 2 (c) 12 (d) none of these

Class work Problem Answers

1. b	2. b	3. a	4. c	5. b	6. c	7. b	8. b	9. a	10.a
11.d	12.c	13.b	14.a	15.c	16.b	17.a	18.b	19.d	20.b
21.a	22.c	23.a	24.c	25.b	26.a	27.c	28.b	29.d	30.c
31.a	32.b	33.a	34.b	35.c	36.b	37.c			

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Homework Problems

1. $\lim_{x\to 3}\frac{x^2+2x-15}{x^2-5x+6}=$

(a) 4 (b) 6 (c) 8 (d) 0

2. $\lim_{x\to a}\frac{x^7-a^7}{x^5-a^5}=$

(a) $\frac{7}{5}a$ (b) $\frac{7}{5}a^2$ (c) $\frac{3}{5}a^2$ (d) none of these

3. $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x}=$

(a) 0 (b) 2 (c) $\frac{1}{2}$ (d) none of these

4. $\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3-x}}{x}$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) 1

5. $\lim_{x\to 3}\frac{(x-3)\left(x^2-2\right)}{x^2-2x-3}=$

(a) $\frac{3}{2}$ (b) $\frac{7}{4}$ (c) $\frac{7}{2}$ (d) $-\frac{7}{4}$

6. $\lim_{x\to 0}\frac{\sqrt{1+x+x^3}-1}{x}=$

(a) $\frac{1}{2}$ (b) 0 (c) 3 (d) 1

7. $\lim_{x\to 2}\frac{\left(x^2+5\right)^{\frac{1}{2}}-3}{x-2}$ equals

(a) 0 (b) $-\frac{\left(\sqrt{5}-3\right)}{2}$ (c) $\frac{2}{3}$ (d) none of these

8. $\lim_{x\to 0}\frac{\sin x^\circ}{x}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{180}$ (c) $\frac{\pi}{2}$ (d) $\frac{180}{\pi}$

9. $\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{4}\right)}{x^2}=$

(a) 4 (b) $\frac{1}{4}$ (c) 16 (d) $\frac{1}{16}$

10. $\lim_{x\to 0}\frac{\sin^2 x/3}{x^2}$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$ (c) 9 (d) 0

11. $\lim_{x\to 1}\left[\frac{\sin \pi x}{x-1}\right]$ is equal to

(a) π (b) 2π (c) $-\pi$ (d) $\frac{-\pi}{2}$

12. $\lim_{x\to 0}\frac{\tan\left[\sin^{-1}3x\right]}{\sin^{-1}(2\tan 2x)}=$

(a) $\frac{3}{4}$ (b) $\frac{3}{2}$ (c) $\frac{4}{3}$ (d) none of these

13. $\lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}=$

(a) $\sqrt{2}$ (b) 2 (c) 0 (d) $\frac{1}{2}$

14. $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)=$

(a) -2 (b) 0 (c) -1 (d) 1

15. The value of $\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$ is

(a) 2 (b) $\frac{1}{4}$ (c) 0 (d) $\frac{1}{2}$

16. $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 0 (d) 1

17. $\lim_{x\to 0}\frac{\log\cos x}{x}$ is equal to

(a) 0 (b) ∞ (c) 1 (d) none of these

18. $\lim_{x\to\frac{\pi}{2}}\frac{\left\{1-\tan\left(\frac{x}{2}\right)\right\}(1-\sin x)}{\left(1+\tan\frac{x}{2}\right)(\pi-2x)^3}$ is

(a) ∞ (b) $\frac{1}{8}$ (c) 0 (d) $\frac{1}{32}$

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19. $\lim_{x\to 0} \frac{3^x - 2^x}{\sin x} =$

(a) $\log\frac{3}{2}$ (b) $\log 1$ (c) $\log 6$ (d) none of these

20. $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$ is equal to

(a) e (b) $e^{1/2}$ (c) e^{-2} (d) none of these

21. $\lim_{n\to\infty}\left[\frac{1+2+3+..........+n}{3n^2+5}\right] =$

(a) 1/6 (b) 1/3 (c) 3 (d) 6

22. $\lim_{x\to 0}(1+ax)^{b/x} =$

(a) e^{ab} (b) e^{a^b} (c) e^{a+b} (d) e

23. $\lim_{n\to\infty}\frac{1^2+2^2+3^2+\ldots+n^2}{n^3} =$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) 0

24. $\lim_{n\to\infty}\frac{1+2+\ldots+n}{n^2+100} =$

(a) $\frac{1}{2}$ (b) 0 (c) 2 (d) ∞

25. $\lim_{n\to\infty}\left(3^n+4^n\right)^{\frac{1}{n}} =$

(a) 4 (b) 3 (c) e^a (d) 1

26. $\lim_{x\to 1}\left(\log_2 2x\right)^{\frac{1}{\log_2 x}} =$

(a) e (b) $\frac{1}{e}$ (c) $2e$ (d) none of these

27. The function $f(x) = |x|$ is continuous at

(a) $(x=0)$ only (b) $x=1$ only (c) all real points (d) none of these

28. If $f(x) = 2x^2+5x+2$ for $x \neq -2$ and $f(-2) = k$, then the value of k is

(a) 2 (b) -2 (c) 1 (d) 0

29. If $f(x)=\frac{x-4}{x-1}$, then $f(x)$ is

(a) continuous at $x=2$ (b) discontinuous at $x=2$

(c) discontinuous at $x=0$ (d) discontinuous at $x=4$

30. $f(x)=\sin x+\log_e x$ is continuous at

(a) all x (b) all $x>0$ (c) all $x<0$ (d) none of these

31. Which one of the following is a false statement?

(a) If $f(x)$ is continuous at $x=a$, then $\lim_{x\to a} f(x)$ exits

(b) If $f'(a)$ exists, then $f(x)$ is continuous at $x=a$

(c) If $f(x)$ is continuous at $x=a$, then $\lim_{x\to a} f(x)=f(a)$

(d) If $\lim_{x\to a} f(x)$ exists, then $f(x)$ is not continuous at $x=a$

32. If $f(x)=\begin{cases}\frac{\sqrt{1+Px}-\sqrt{1-Px}}{x}, & -1\le x<0\\ \frac{2x+1}{x-2}, & 0\le x\le 1\end{cases}$ is continuous in the interval $[-1,1]$ then P is equal to

(a) -1 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1

33. $f(x)=\begin{cases}-1 & : & x<-1\\ -x & : & -1\le x\le 1\\ 1 & : & x>1\end{cases}$ is continuous

(a) at $x=1$ but not at $x=-1$ (b) at $x=-1$ but not at $x=1$

(c) at both $x=1$ and $x=-1$ (d) neither at $x=1$ nor at $x=-1$

34. If $f(x)=\begin{cases}\frac{x^5-32}{x-2}, & x\ne 2\\ K, & x=2\end{cases}$ is continuous at $x=2$, then $K=$

(a) 16 (b) 32 (c) 80 (d) 8

35. If $f(x)=\begin{cases}\frac{1-\cos x}{x}, & x\ne 0\\ k, & x=0\end{cases}$ is continuous at $x=0$, then $k=$

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{2}$

36. The function $f(x)=\begin{cases} x^2/a & 0\le x\le 1 \\ a & 1\le x<\sqrt{2} \\ \frac{(2b^2-4b)}{x^2} & \sqrt{2}\le x<\infty \end{cases}$ is continuous for $0\le x<\infty$, then the most suitable values of a and b are

(a) $a=1, b=-1$ (b) $a=-1, b=1+\sqrt{2}$

(c) $a=-1, b=1$ (d) None of these

37. $\lim_{x\to 0}\log_{\cos\frac{x}{2}}(\cos x)=$

(a) 1 (b) 2 (c) 3 (d) 4

Homework Problem Answers

1. c	2. b	3. a	4. a	5. b	6. a	7. c	8. b	9. d	10.b
11.c	12.b	13.c	14.b	15.b	16.a	17.a	18.d	19.a	20.d
21.a	22.a	23.c	24.a	25.a	26.a	27.a	28.d	29.a	30.b
31.d	32.b	33.d	34.c	35.a	36.c	37.d			